Equation (11), together with Eqs. (10) and (12), gives the solution for Eq. (1).

An example with nonlinearity parameter $\bar\epsilon = 0.2$ is worked out for various values of $\tau/T$. In Fig. 1, time-displacement relations are plotted for nonresonance solutions for various values of $\tau/T$, with ultraspherical polynomial index $\lambda = 1/2$. It is seen that the method gives results which compare well with numerical solution, obtained by using a Runge-Kutta fourth-order algorithm on the PACER 600 computer. The linear solution is also plotted in Fig. 1.

### Limitation of the Approximation Method

The approximation method fails to give accurate results for resonance solutions. For the values of $\tau/T = 3/8$, $1/2$, and $5/8$, it fails to give even qualitative results; for $\tau/T = 3/4$ and $7/8$, it gives quantitatively wrong results. In Fig. 2, time-displacement relations are plotted for resonance solutions for various values of $\tau/T$.

In Fig. 3, first peak and time taken to reach first peak are plotted vs $\tau/T$. For nonresonance solutions, results are taken from approximate solution and for resonance solutions, it is taken from numerical solution. It is seen that maximax response occurs at about $\tau/T = 0.7$.

### Conclusions

1) The method of ultraspherical polynomial approximation gives good results for nonresonance solutions.
2) It completely fails at and around resonance, that is, at $\tau/T = 0.5$.
3) The maximax response occurs at about $\tau/T = 0.7$.

### References


[1] Denman, H. H., "Amplitude Dependence of Frequency in a Linear Approximation to the Simple Pendulum Equation," *American Journal of Physics*, Vol. 27, Oct. 1959, pp. 524-525.

[2] Denman, H. H. and Howard, J. E., "Application of Ultraspherical Polynomials. I-Free Oscillation of the Pendulum," *Quarterly of Applied Mathematics*, Vol. 21, Jan. 1964, pp. 325-330.

[3] Denman, H. H. and Liu, Y. K., "Application of Ultraspherical Polynomials to Nonlinear Oscillations. II-Free Oscillations," *Quarterly of Applied Mathematics*, Vol. 22, Jan. 1965, pp. 273-292.

[4] Jacobsen, L. S. and Ayre, R. S., *Engineering Vibrations*, McGraw-Hill, New York, 1958, pp. 135-145.

[5] Anderson, G. L., "An Approximate Analysis of Nonlinear Non-conservative Systems Using Orthogonal Polynomials," *Journal of Sound and Vibration*, Vol. 29, Aug. 1973, p. 463-474.


J80-~~[illegible]~~ 220

# Low-Frequency and Small Perturbation Equation for Transonic Flow Past Wings

20018


S. K. Chakrabartty* and N. R. Subramanian*
*National Aeronautical Laboratory, Bangalore, India*


### Introduction

MOST methods for transonic flow calculations for arbitrary lifting wings and moderately complicated wing-fuselage combinations use transonic small perturbation theory, but differ in model equation.[1] Their shock relations have been investigated in Ref. 1 and compared with that of full potential equations and the Rankine-Hugoniot relation. It is apparent from the comparison that the equation used in NLR is the best of the methods for the calculation of flows with shocks past finite wings. Couston and Angelini[2] presented an original approach to derive a two-dimensional, low-frequency small perturbation equation and its corresponding boundary condition which avoids some of the arbitrariness of the equation used in Ref. 3 by using the concept of weak solution. Recently, Schmidt[4] formulated a self-consistent transonic small disturbance equation and boundary condition along the same line as that in Ref. 1. Making the approximations at the functional level and using the concept of weak solution as was done in Ref. 2 for a two-dimensional case, the present Note gives the small perturbation equations and the boundary conditions for moderate aspect ratio finite wings. The corresponding modifications for large aspect ratio and low aspect ratio slender wings can be carried out from the order of magnitude of different terms in different cases.

### Derivation of Small Perturbation Equations

Consider a three-dimensional unsteady potential flow past an arbitrary wing in a Cartesian coordinate system $X, Y$, and $Z$. The $X$ axis coincides with the freestream direction $Y$ along the spanwise direction, and $Z$ is perpendicular to the $X$-$Y$ plane to form a right-handed system with the origin at the leading edge of the root chord. The equation for the conservation of mass in terms of the velocity potential $\Phi$ and density $\bar\rho$ can be written as

$$\frac{\partial\bar\rho}{\partial\bar t} + \mathrm{div}\,(\bar\rho\,\overrightarrow{\mathrm{grad}}\Phi) = 0 \tag{1}$$

The Bernouilli equation gives

$$\frac{\partial\Phi}{\partial\bar t} + \frac{1}{2}[\overrightarrow{\mathrm{grad}}^2\Phi - U_\infty^2] + \frac{(\bar\rho/\rho_\infty)^{\gamma-1}-1}{\gamma-1}c_\infty^2 = 0 \tag{2}$$

where $\gamma$ is the ratio of the specific heats, $c_\infty$ is the speed of sound at infinity, $U_\infty$ is the freestream velocity, and $\bar t$ is the time.

The desired solution should have the property that $\Phi$ and its first derivatives are piecewise continuous, satisfying the mass conservation law [see Eq. (1)] at points where the flow is smooth together with a jump condition across a discontinuity. That is to say, $\Phi$ should be the weak solution of Eq. (1). Interest is restricted to the solution of periodic flow motion with small amplitude. For any periodic test function $\psi$, the weak equation corresponding to Eq. (1) is written

$$\iiiint\left(\bar\rho\frac{\partial\psi}{\partial\bar t} + \bar\rho\,\overrightarrow{\mathrm{grad}}\Phi\cdot\overrightarrow{\mathrm{grad}}\psi\right)\mathrm{d}X\mathrm{d}Y\mathrm{d}Z\mathrm{d}\bar t = 0 \tag{3}$$

The solution of Eq. (3) will satisfy Eq. (1) in the weak sense and the jump conditions across any discontinuity.[5] Consider the functional

$$\mathcal{L}(\Phi) = \iiiint \frac{(\bar\rho/\rho_\infty)^{\gamma}-1}{\gamma M_\infty^2}\mathrm{d}X\mathrm{d}Y\mathrm{d}Z\mathrm{d}\bar t \tag{4}$$

where, from Eq. (2),

$$(\bar\rho/\rho_\infty)^{\gamma} = \left\{1 - \frac{\gamma-1}{c_\infty^2}\left[\frac{\partial\Phi}{\partial\bar t} + \frac{1}{2}(\overrightarrow{\mathrm{grad}}^2\Phi - U_\infty^2)\right]\right\}^{\frac{\gamma}{\gamma-1}} \tag{5}$$

It can be shown that the Gateaux-difference

$$\delta\mathcal{L}(\Phi) = \lim_{\bar\lambda\to 0}\left[\frac{\mathcal{L}(\Phi+\bar\lambda\psi) - \mathcal{L}(\Phi-\bar\lambda\psi)}{2\bar\lambda}\right] \tag{6}$$

Received Oct. 25, 1979; revision received Jan. 7, 1980. 

Index category: Transonic Flow.

*Scientist, Aerodynamics Division.

corresponds to the left-hand side of Eq. (3). Then, the problem becomes

$$\delta\mathcal{L}(\Phi)=0 \tag{7}$$

It is known that the best approximations are often obtained at the level of the functional. Since a small perturbation approach is of interest here, a perturbation potential $\phi$ is defined as

$$\Phi=U_\infty x+\phi \tag{8}$$

The velocity components are now

$$u=U_\infty+\phi_X,\quad v=\phi_Y,\quad \omega=c_Z \tag{9}$$

such that $\phi_X$, $\phi_Y$, $\phi_Z$ are all small.

Define a typical wing chord of length unity, such that the condition $\phi_X \ll 1$ implies $\phi \ll 1$. Introduce two lengths $b$ and $c$ in the $Y$ and $Z$ directions, respectively, and the new nondimensional quantities as

$$X=x,\;\; Y=by,\;\; Z=cz,\; \phi=U_\infty\varphi,\;\; t=U_\infty K\bar{t},\;\; \rho=\bar{\rho}/\rho_\infty \tag{10}$$

where $K=w/U_\infty$ is the reduced frequency.

Equation (5) can be written in the form

$$\rho^\gamma=(1+\sigma)^{\gamma/\gamma-1} \tag{11}$$

Expanding Eq. (11) in a Taylor series with respect to the small quantity $\sigma$, making use of the conditions

$$\left.\begin{aligned}\varphi&\sim\epsilon\ll 1\\ \varphi_y&\sim\epsilon/b\ll 1\\ \varphi_z&\sim\epsilon/c\ll 1\\ 1-M_\infty^2&\sim\epsilon\ll 1\end{aligned}\right\} \tag{12}$$

and retaining the terms up to second-order,[6] Eq. (7) becomes

$$\begin{aligned}\delta\iiiint\Big[&-K\frac{\partial\varphi}{\partial t}-\frac{\partial\varphi}{\partial x}+KM_\infty^2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial t}\\ &-\frac{1}{2}(1-M_\infty^2)\left(\frac{\partial\varphi}{\partial x}\right)^2-\frac{1}{2}\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\left\{\left(\frac{\partial\varphi}{\partial y}\right)^2\right.\\ &\left.+\left(\frac{\partial\varphi}{\partial z}\right)^2\right\}+\frac{1}{6}\{(\gamma+1)M_\infty^2+3(1-M_\infty^2)\}M_\infty^2\\ &\times\left(\frac{\partial\varphi}{\partial x}\right)^3\Big]\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t=0\end{aligned} \tag{13}$$

assuming the reduced frequency $K$ to be small.

Calculating the Gateaux-difference [see Eq. (6)] and going back through all the processes up to Eq. (3),

$$\begin{aligned}\iiiint\Big[&\frac{\partial}{\partial t}\left\{K\psi\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\right\}\\ &+\frac{\partial}{\partial x}\left[\psi\left\{1+(1-M_\infty^2)\frac{\partial\varphi}{\partial x}-\frac{\lambda}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2-KM_\infty^2\frac{\partial\varphi}{\partial t}\right.\right.\\ &\left.\left.-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial y}\right)^2-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial z}\right)^2\right\}\right]\\ &+\frac{\partial}{\partial y}\left[\psi\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial y}\right]\\ &+\frac{\partial}{\partial z}\left[\psi\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial z}\right]\Big]\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\ -\iiiint\psi\Big[&-2KM_\infty^2\frac{\partial^2\varphi}{\partial x\partial t}+(1-M_\infty^2)\frac{\partial^2\varphi}{\partial x^2}\\ &-\lambda\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}-2M_\infty^2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}-2M_\infty^2\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial x\partial z}\\ &+\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial y^2}+\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial z^2}\Big]\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t=0\end{aligned} \tag{14}$$

is reached with

$$\lambda=[(\gamma+1)M_\infty^2+3(1-M_\infty^2)]M_\infty^2 \tag{15}$$

If the region of the flow is bounded by a surface at infinity and the surfaces defined by $B(x,y,z,t)=0$, then the first integral can be converted to a surface integral by using the divergence formula. Now, the integrands are the product of the test function $\psi$. The small perturbation form of the true potential $\Phi$ will satisfy two integrals simultaneously for the arbitrary test function $\psi$. So, $\varphi$ should satisfy

$$\begin{aligned}2KM_\infty^2\frac{\partial^2\varphi}{\partial x\partial t}&=(1-M_\infty^2)\frac{\partial^2\varphi}{\partial x^2}-\lambda\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}\\ &-2M_\infty^2\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}-2M_\infty^2\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial x\partial z}\\ &+\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial y^2}+\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial z^2}\end{aligned} \tag{16}$$

from the second term and

$$\begin{aligned}&\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)B_t+B_x\left\{1+(1-M_\infty^2)\frac{\partial\varphi}{\partial x}-\frac{\lambda}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right.\\ &\left.-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial y}\right)^2-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial z}\right)^2\right\}\\ &+B_y\left\{\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial y}\right\}+B_z\left\{\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial z}\right\}=0\end{aligned} \tag{17}$$

from the first term.

For a three-dimensional wing, the surface $B(x,y,z,t)=0$ consists of the wing surface $H(x,y,z,t)=0$, the wake $W(x,y,z,t)=0$, and the shock $S(x,y,z,t)=0$. So from Eq. (17), the boundary condition on the wing surface becomes

$$\begin{aligned}&\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial z}-\tan\alpha\left[1+(1-M_\infty^2)\frac{\partial\varphi}{\partial x}\right.\\ &\left.-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial y}\right)^2-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial z}\right)^2-\frac{\lambda}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\\ &-\tan\beta\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial y}=K\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial h}{\partial t}\end{aligned} \tag{18}$$

where $\tan\alpha$ and $\tan\beta$ are local instantaneous airfoil slopes in the $x$ and $y$ directions, respectively, and $H=z-h(x,y,t)$ is the function describing the wing surface.

If we assume the mean position of the wake to be on the plane $z=0$, then from Eq. (17) it follows

$$\begin{aligned}&\left\langle\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial z}\right\rangle_{\text{across the wake}}\\ &=\left\langle\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)W_t\right\rangle_{\text{across the wake}}\end{aligned} \tag{19}$$

where the symbol $\langle\;\rangle$ indicates the jump.

Since there is no pressure load on the wake and $\langle W_t\rangle=0$, $\partial\varphi/\partial z$ is continuous across the wake.

Similarly, if $\cos\alpha_s$, $\cos\beta_s$ and $\cos\gamma_s$ are the direction cosines of the shock surface with respect to the $x$, $y$, and $z$ axes, respectively, then the jump condition across the shock becomes

$$\left\langle\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)S_t\right\rangle+\cos\alpha_s\left\langle 1+(1-M_\infty^2)\frac{\partial\varphi}{\partial x}\right.$$
$$\left.-\frac{\lambda}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial y}\right)^2-\frac{1}{2}M_\infty^2\left(\frac{\partial\varphi}{\partial z}\right)^2\right\rangle$$
$$+\cos\beta_s\left\langle\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial y}\right\rangle+\cos\gamma_s\left\langle\left(1-M_\infty^2\frac{\partial\varphi}{\partial x}\right)\frac{\partial\varphi}{\partial z}\right\rangle=0 \tag{20}$$

The equations previously derived are for general three-dimensional finite wings.

Equation (16) can be further simplified for wings of aspect ratio of the order of one, or of any aspect ratio but appreciable sweep. Following exactly the analysis given by Hall and Firmin,[6]

$$2KM_\infty^2\frac{\partial^2\varphi}{\partial x\partial t}=[1-M_\infty^2-\lambda\varphi_x]\varphi_{xx}$$
$$+\varphi_{yy}+\varphi_{zz}-2\varphi_y\varphi_{xy}-2\varphi_z\varphi_{xz} \tag{21}$$

For $b\rightarrow\infty$, i.e., for a flow approaching that past a two-dimensional airfoil, $\varphi_{yy}\rightarrow 0$ and $\varphi_y\varphi_{xy}\rightarrow 0$, while $\varphi_z\varphi_{xz}$ $(\sim\epsilon^2/c^2)\ll\varphi_{zz}(\sim\epsilon/c^2)$ and the equation reduces to

$$[1-M_\infty^2-\lambda\varphi_x]\varphi_{xx}+\varphi_{zz}=2KM_\infty^2\varphi_{xt} \tag{22}$$

Equation (21) is the same as was taken by Hall and Firmin[6] except for the value of $\lambda$, where it was taken as

$$\lambda=(\gamma+1)M_\infty^2 \tag{23}$$

The Eq. (22) has been studied in Ref. 2. The steady-state equation corresponding to Eq. (16) with the same value of $\lambda$ has been studied in NLR (Ref. 1), where the cross terms $\varphi_z\varphi_{xz}$ and $\varphi_x\varphi_{zz}$, which are permissible for large aspect ratio wings or wings with very low sweep have been neglected.

## Acknowledgment

The authors are thankful to Dr. U. N. Sinha of NAL for helpful discussions.

## References

[1] Vooren, J.V.D., Sloof, J. W., Huizing, G. H., and Essen, A. V., "Remarks on the Suitability of Various Transonic Small Perturbation Equations to Describe Three-Dimensional Transonic Flow: Examples of Computations Using a Fully Conservative Rotated Difference Scheme," edited by K. Oswatitsch and D. Rues, *Symposium Transsonicum II*, Springer-Verlag, New York, 1975, pp. 557-566.

[2] Couston, P. M. and Angelini, J. J., "Solution of Nonsteady Two-Dimensional Transonic Small Disturbance Potential Flow Equations," ONERA TP 1978-69.

[3] Ballhaus, W. F., "Some Recent Progress in Transonic Flow Computation," *VKI Lecture Series—Computational Fluid Dynamics*, Von Kármán Institute for Fluid Dynamics, Rhode-St. Genese, Belgium, March 1976.

[4] Schmidt, W., "A Self-Consistent Formulation of the Transonic Small Disturbance Theory," *Recent Developments in Theoretical and Experimental Fluid Mechanics*, edited by U. Muller et al., Springer-Verlag, New York, 1979, pp. 48-57.

[5] Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Vol. II, Wiley Interscience, New York, 1962.

[6] Hall, M. G. and Firmin, M.C.P., "Recent Developments in Methods for Calculating Transonic Flows over Wings," edited by R. R. Dexter and J. Singer, *ICAS Proceedings*, The Weizmann Science Press of Israel, Jerusalem, Vol. 1, 1974, pp. 134-151.

J80-~~[illegible]~~ 221

# Similarity Rule for Sidewall Boundary-Layer Effect in Two-Dimensional Wind Tunnels

20016 20018 30016

Richard W. Barnwell*
*NASA Langley Research Center, Hampton, Va.*

## Introduction

THE interference which the walls of a wind tunnel exert on the flow in the tunnel has long been a subject of concern and study. In the case of the two-dimensional or nearly two-dimensional wind tunnels which are used to test airfoils, the interference effects caused by the sidewalls are substantially different from those caused by the upper and lower walls.

The interference caused by the upper and lower walls of two-dimensional wind tunnels is primarily inviscid. The principal modification made to the upper and lower walls to relieve interference is ventilation with holes (pores) or longitudinal slots to relieve blockage effects. A summary of analytical methods which have been developed to predict blockage and lift interference effects in subsonic tunnels with closed, open, and ventilated upper and lower walls is given in Ref. 1.

The origin of sidewall interference in two-dimensional wind tunnels is viscous. The two sidewall interference problems which have received the most attention are the flat-plate-type growth of the sidewall boundary layer and the separation of the sidewall boundary layer due to large model-induced pressure gradients. Very little attention has been paid to the intermediate problem of the interaction of attached sidewall boundary layers with model-induced pressure gradients. Recently, experimental results were presented in Refs. 2 and 3 which can be used to evaluate this effect quantitatively. In the present paper, a simple analysis is presented which results in a similarity rule that relates compressibility and the interaction effect of the sidewall boundary layer to the model-induced pressure field. It is shown that this similarity rule is consistent with the relevant data in Refs. 2 and 3. An earlier version of the present study is given in Ref. 4.

## Analysis

Consider steady, isentropic, small-perturbation flow in a nominally two-dimensional airfoil wind tunnel. Let the Cartesian coordinates in the freestream, normal, and spanwise directions be $x$, $y$, and $z$, and the respective velocity components be $u$, $v$, and $w$. The effective tunnel width is $b-2\delta^*$ where $b$ and $\delta^*$ are the tunnel width and the sidewall displace thickness. It is assumed that $\delta^*$ can vary slightly with respect to $x$ and $y$, and that the boundary conditions for the airfoil model and the upper and lower walls are independent of $z$. It is also assumed that the tunnel is narrow enough for flow at each sidewall to be strongly influenced by the other sidewall boundary layer. To lowest order, the spanwise velocity component in this tunnel varies linearly with the spanwise coordinate $z$ as

$$w=-\frac{2uz}{b}\frac{\partial\delta^*}{\partial x} \tag{1}$$

Presented as Paper 79-0108 at the AIAA 17th Aerospace Sciences Meeting, New Orleans, La., Jan. 15-17, 1979; submitted Feb. 12, 1979; revision received Feb. 29, 1980. This paper is declared a work of the U.S. Government and therefore is in the public domain.

Index categories: Research Facilities and Instrumentation; Subsonic Flow; Transonic Flow.

*Head, Transonic Aerodynamics Branch.